PHYS 301 - Tutorial 3 - Sept. 22, 2024

Group Problem Sol'ns

1. (a)

$$I = \int_V (r^4 - 1)\,\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) d\tau$$

Applying integration by parts requires:

$$\int_V f(\vec{\nabla}\cdot\vec{A})\,d\tau = \oint_S f\vec{A}\cdot d\vec{a} - \int_V \vec{A}\cdot\vec{\nabla}f\,d\tau$$

Here, $f = r^4 - 1 \Rightarrow \vec{\nabla}f = 4r^3\hat{r}$

$\vec{A} = \dfrac{\hat{r}}{r^2}$

$$\therefore I = \underbrace{\oint_S (r^4-1)\frac{\hat{r}}{r^2}\cdot d\vec{a}}_{I_1} - \underbrace{\int_V \frac{\hat{r}}{r^2}\cdot(4r^3\hat{r})\,d\tau}_{I_2}$$

$d\vec{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

$d\tau = r^2 \sin\theta\, d\theta\, d\phi\, dr$

$\therefore (r^4 - 1)\dfrac{\hat{r}}{r^2} \cdot d\vec{a}$ becomes

$$(R^4 - 1)\frac{1}{R^2} R^2 \sin\theta \, d\theta \, d\phi = (R^4 - 1) \sin\theta \, d\theta \, d\phi$$

$$I_1 = (R^4 - 1) \underbrace{\int_{\theta=0}^{\pi} \sin\theta \, d\theta}_{2} \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi} = 4\pi (R^4 - 1)$$

$$I_2 = \int_V 4r \, r^2 \sin\theta \, d\theta \, d\phi = 16\pi \int_{r=0}^{R} r^3 \, dr$$

$$= 4\pi R^4$$

$$\therefore I = I_1 - I_2 = 4\pi(\cancel{R^4} - 1) - \cancel{4\pi R^4}$$

$$\boxed{\therefore I = -4\pi}$$

Alternative sol'n using the fact that $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})$

$$I = \int_V (r^4-1)\,4\pi\,\delta^3(\vec{r})\,d\tau = 4\pi(0^4-1)$$

$$= \boxed{-4\pi} \quad \text{same.} \checkmark$$

(b) Prove that:

$$\int_S f(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint_P f\vec{A}\cdot d\vec{l} + \int_S (\vec{A}\times\vec{\nabla}f)\cdot d\vec{a}$$

Start w/ product rule (7) from front cover of Griffiths:

$$\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$$

$$\therefore f(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\times(f\vec{A}) + \vec{A}\times(\vec{\nabla}f)$$

$$\therefore \int_S f(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int_S \vec{\nabla}\times(f\vec{A})\cdot d\vec{a} + \int_S \vec{A}\times(\vec{\nabla}f)\cdot d\vec{a}$$

Now apply Stoke's theorem to first term on RHS.

$$\int_S \vec{\nabla} \times \vec{v} \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$$

In our problem, $\vec{v} = f\vec{A}$

$$\boxed{\therefore \int_S f(\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint_P f\vec{A} \cdot d\vec{l} + \int (\vec{A} \times \vec{\nabla} f) \cdot d\vec{a}}$$

2. Start w/ $I = \int_{-a}^{b} f(x)\, x \frac{d\,\delta(x)}{dx}\, dx$

and integrate by parts : $\int_{x_1}^{x_2} u\, dv = uv \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} v\, du$

In our problem, $dv = \frac{d\,\delta(x)}{dx}\, dx \longrightarrow v = \delta(x)$

$u = x f(x) \longrightarrow du = x \frac{df(x)}{dx} + f(x)$

$$\therefore I = \underbrace{x f(x)\, \delta(x) \Big|_{-a}^{b}}_{\text{zero b/c } \delta(b) = \delta(-a) = 0.} - \int_{-a}^{b} \delta(x) \left[x \frac{df(x)}{dx} + f(x) \right] dx$$

$$\therefore I = \underbrace{- \int_{-a}^{b} \delta(x)\, x \frac{df(x)}{dx}\, dx}_{= x \frac{df(x)}{dx} \Big|_{x=0} = 0} - \int_{-a}^{b} \delta(x) f(x)\, dx$$

$$\therefore \int_{-a}^{b} f(x)\, x\, \frac{d\delta(x)}{dx}\, dx = -\int_{-a}^{b} \delta(x)\, f(x)\, dx$$

Integrands on left & right must be equal,

$$\therefore \boxed{x \frac{d\delta(x)}{dx} = -\delta(x)}$$

(b)

Heaviside fcn.

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Start w/ $$J = \int_{-\infty}^{\infty} f(x)\, \frac{d\theta(x)}{dx}\, dx$$

$$u = f(x) \longrightarrow du = \frac{df(x)}{dx}\, dx$$

$$dv = \frac{d\theta(x)}{dx}\, dx \longrightarrow v = \theta(x)$$

$$J = f(x)\theta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \theta(x)\,\frac{df(x)}{dx}\,dx$$

$$= f(\infty) - \underbrace{\int_{0}^{\infty} \frac{df(x)}{dx}\,dx}_{\substack{f(\infty)-f(0) \\ \text{by Fundamental Th.} \\ \text{of calculus}}}$$

since $\theta(x) = 0$ for $x < 0$

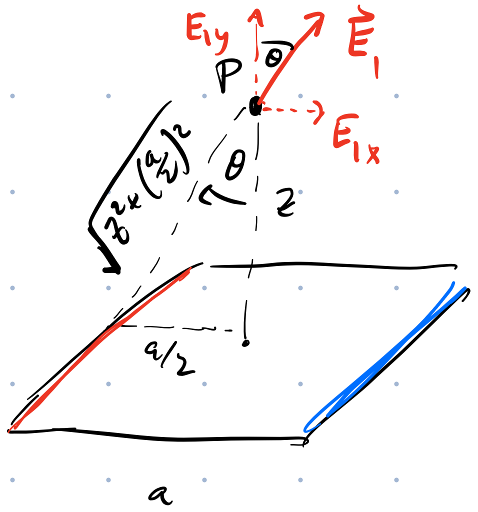
$$\therefore J = \cancel{f(\infty)} - \left(\cancel{f(\infty)} - f(0)\right) = f(0)$$

by $f(0) = \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = J$

$$\therefore \int_{-\infty}^{\infty} f(x)\,\frac{d\theta(x)}{dx}\,dx = \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$$

$$\therefore \boxed{\delta(x) = \frac{d\theta(x)}{dx}}$$

3.



Notice that distance from midpt of one wire to P is $\sqrt{z^2+\left(\frac{a}{2}\right)^2}$

The red wire makes $\vec{E}_1$ but, by symmetry, only its vertical component will survive. The horizontal component will cancel w/ that of blue wire.

$$E_{1y} = E_1 \cos\theta$$

where $\cos\theta = \dfrac{z}{\sqrt{z^2+\left(\frac{a}{2}\right)^2}}$

There will be 4 such contributions at P due to the 4 wires.

$$\therefore \quad \vec{E} = 4E_1 \frac{z}{\sqrt{z^2+\left(\frac{a}{2}\right)^2}} \hat{z}$$

but $E_1 = \dfrac{1}{4\pi\varepsilon_0} \dfrac{\lambda a}{\sqrt{z^2+\left(\frac{a}{2}\right)^2}\,\sqrt{z^2+\left(\frac{a}{2}\right)^2+\left(\frac{a}{2}\right)^2}}$

(using the given expression for $\vec{E}$ after taking $z \to \sqrt{z^2+\left(\frac{a}{2}\right)^2}$)

$$\boxed{\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{4\lambda a z}{\left[z^2+\left(\frac{a}{2}\right)^2\right]\sqrt{z^2+\frac{a^2}{2}}}\,\hat{z}}$$

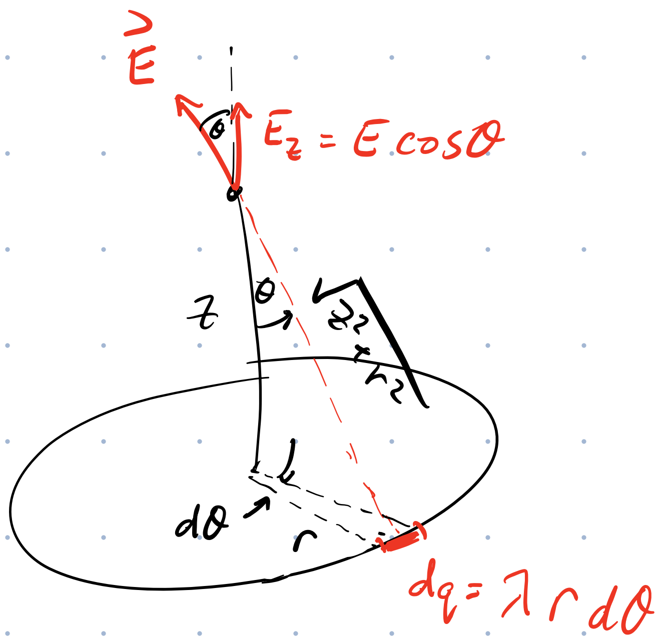
If $z \to \infty$, then $z^2 \gg a^2$

$$\vec{E} \approx \frac{1}{4\pi\varepsilon_0} \frac{(4a)\lambda\, \cancel{z}}{(z^2)\,(\cancel{z})}\,\hat{z}$$

but $4a\lambda$ is the total charge of the square loop, say Q_{sq}.

$\therefore \vec{E} = \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{z^2}\,\hat{z}$ $\rightarrow$ like a pt. charge ✓

4(a)



$$\cos\theta = \frac{z}{\sqrt{z^2+r^2}}$$

like in the previous problem, only the z-components of $\vec{E}$ survive.

dE_z @ P due to dq:

$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{dq}{z^2+r^2}\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda r\, d\theta\ z}{(z^2+r^2)^{3/2}}$$

$$\therefore E_z = \int dE_z = \int_{\theta=0}^{2\pi}\frac{1}{4\pi\varepsilon_0}\frac{\lambda r z}{(z^2+r^2)^{3/2}}\,d\theta$$

Everything is a const, except $d\theta$

$$\boxed{\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\pi r\lambda z}{(z^2+r^2)^{3/2}}\,\hat{z}}$$

For $z \to \infty$ $\quad (z^2+r^2)^{3/2} \approx z^3$

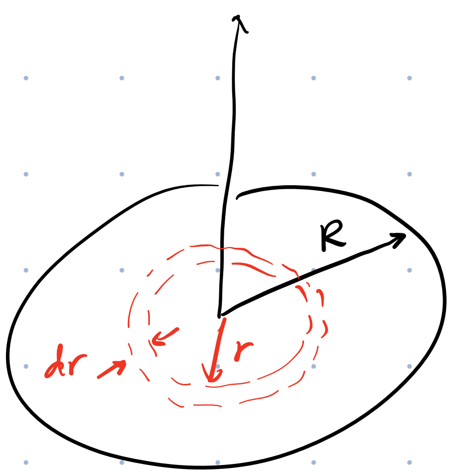
& $\underbrace{2\pi r}_{\text{circumference of loop}} \lambda = q_{loop} \Rightarrow$ the total charge of the loop.

$\therefore$ In $z \to \infty$ limit

$$\vec{E} \approx \frac{1}{4\pi\varepsilon_0} \frac{q_{loop}\, z}{z^3} \hat{z}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q_{loop}}{z^2} \hat{z}$$ like a pt. charge ✓

(b) Can build the disk by adding together a bunch of thin rings.



The charge of one ring is

$$\sigma\, 2\pi r\, dr$$

$$= \lambda\, 2\pi r$$

$\therefore \lambda = \sigma\, dr$

Electric field due to one ring is given by result of (a).

$$dE_z = \frac{1}{4\pi\varepsilon_0} \frac{2\pi r \lambda z}{(z^2+r^2)^{3/2}}$$

sub in $\lambda = \sigma dr$ & integrate.

$$E_z = \int dE_z = \frac{1}{4\pi\varepsilon_0} \int_{r=0}^{R} \frac{2\pi r \sigma z}{(z^2+r^2)^{3/2}} dr$$

sub $u = z^2+r^2$ $\quad du = 2r\,dr$

$r=0, \quad u = z^2$

$r=R, \quad u = z^2+R^2$

$$E_z = \frac{1}{4\pi\varepsilon_0} \int_{u=z^2}^{z^2+R^2} \pi\sigma z u^{-3/2} du$$

$$= \frac{\pi\sigma z}{4\pi\varepsilon_0} (-2) \frac{1}{\sqrt{u}} \Bigg|_{z^2}^{z^2+R^2}$$

$$\therefore E_z = \frac{2\pi\sigma z}{4\pi\varepsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right]$$

$$\boxed{\therefore \vec{E} = \frac{2\pi\sigma z}{4\pi\varepsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right]\hat{z}}$$

In $z \to \infty$ limit, have

$$\vec{E} \approx \frac{2\pi\sigma z}{4\pi\varepsilon_0}\left[\frac{1}{z} - \frac{1}{z}\frac{1}{\sqrt{1+\left(\frac{R}{z}\right)^2}}\right]\hat{z}$$

use binomial expansion to write

$$\left[1+\left(\frac{R}{z}\right)^2\right]^{-1/2} \approx 1 - \frac{1}{2}\left(\frac{R}{z}\right)^2$$

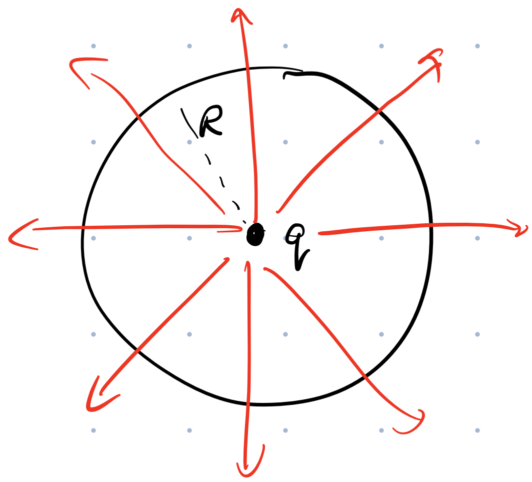
valid for $\left|\frac{R}{z}\right| \ll 1$

$$\therefore \vec{E} \approx \frac{2\pi\sigma z}{4\pi\varepsilon_0}\left[\cancel{\frac{1}{z}} - \frac{1}{z}\left(\cancel{1} - \frac{1}{2}\left(\frac{R}{z}\right)^2\right)\right]\hat{z}$$

$$= \frac{\cancel{2}\pi\sigma \cancel{z}}{4\pi\varepsilon_0}\left[\frac{\cancel{1}}{\cancel{z}}\,\frac{R^2}{z^{\cancel{3}\,2}}\right]\hat{z} = \frac{(\pi R^2\sigma)}{4\pi\varepsilon_0 z^2}\hat{z}$$

but $\pi R^2\sigma = Q_{disk}$ $\quad \therefore \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q_{disk}}{z^2}\hat{z}$ like a pt. charge ✓

5. Start by considering a pt charge q in the centre of a sphere.



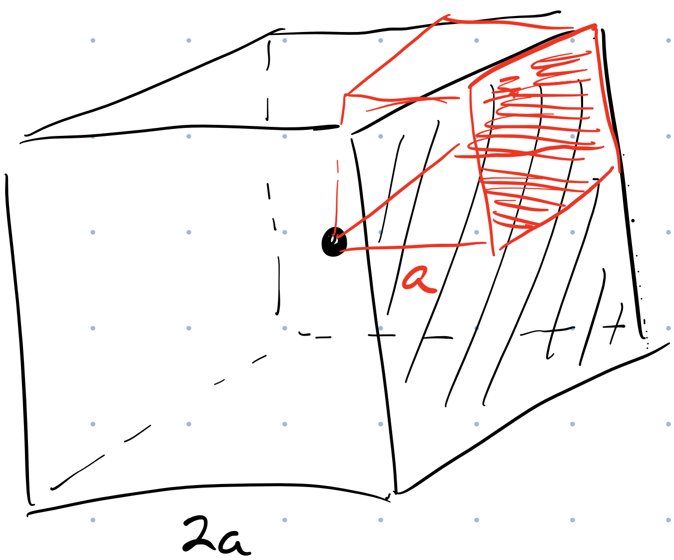
$$\Phi_E = \int \vec{E} \cdot d\vec{a}$$

$$= \int \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{\cancel{R^2}}\hat{r}\right)\cdot\left(\cancel{R^2}\sin\theta\, d\theta\, d\phi\right)$$

$$= \frac{q}{4\pi\varepsilon_0}4\pi = \frac{q}{\varepsilon_0}$$

Φ_E "counts" the number of field lines crossing the surface.

$\therefore$ can change shape w/o changing Φ_E as long as q_{encl} stays fixed. Replace sphere w/ cube of side $2a$.



q still in centre.

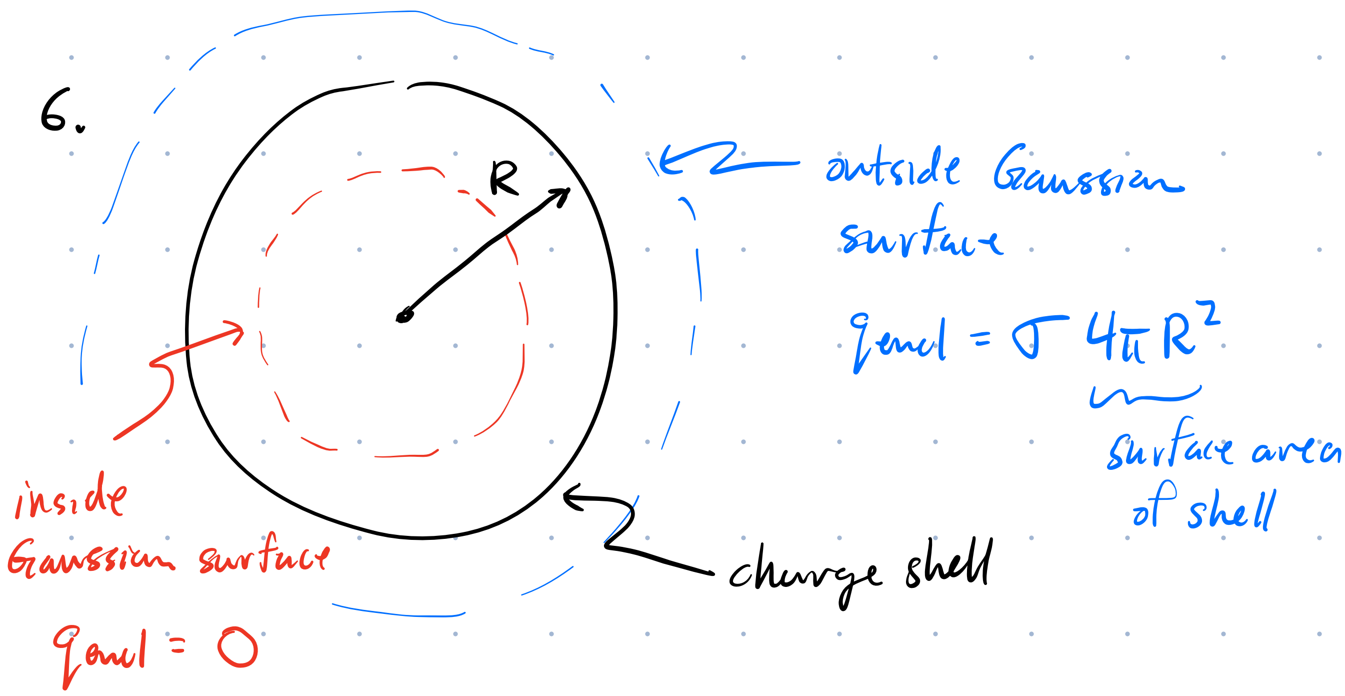
Φ_E is stil $\frac{q}{\varepsilon_0}$. Flux through shaded side is $\frac{1}{6}$ the total.

$$\therefore \Phi_{side} = \frac{q}{6\varepsilon_0}$$

Can fit a cube of side a in large cube w/ q at corner of red square. Flux through red shaded side is clearly $\frac{1}{4}$ flux through black shaded side.

$$\boxed{\therefore \Phi_{red,\, side} = \frac{q}{24\varepsilon_0}}$$

6.



(i) Inside

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{encl}}{\varepsilon_0}$$

but $q_{encl} = 0$ $\boxed{\therefore \vec{E} = 0}$

(ii) Outside

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{encl}}{\varepsilon_0} = \frac{\sigma 4\pi R^2}{\varepsilon_0}$$

by symmetry, $\vec{E}$ is radial $\therefore \vec{E} = E\hat{r}$

$d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

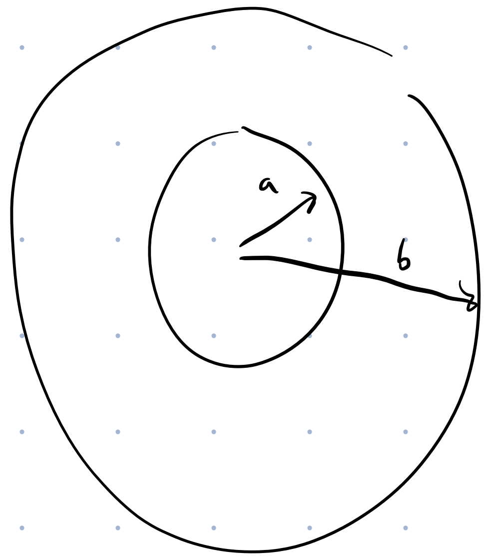
$\Big\} \; \vec{E} \cdot d\vec{a} = E r^2 \sin\theta \, d\theta \, d\phi$

$$\therefore E r^2 \underbrace{\int \sin\theta \, d\theta \, d\phi}_{4\pi} = \frac{\sigma 4\pi R^2}{\varepsilon_0} \Rightarrow E = \frac{1}{4\pi\varepsilon_0} \frac{(\sigma 4\pi R^2)}{r^2}$$

since $\sigma = \dfrac{Q_{shell}}{4\pi R^2}$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q_{shell}}{r^2} \hat{r}$$ like a pt. charge!

7.



(a) (i) For $r < a$, $q_{encl} = 0$ $\boxed{\therefore E = 0}$ like in the previous problem.

(iii) $r > b$

In this case, use a Gaussian surface w/ $r > b$. $\therefore$ q_{encl} is total charge of shell.

$$q_{encl} = \int \rho \, d\tau = \int \frac{k}{\cancel{r^2}} \cancel{r^2} \sin\theta \, d\theta \, d\phi \, dr$$

$$= 4\pi k \int_a^b dr = 4\pi k (b-a)$$

$$\therefore q_{encl} = 4\pi k \left(b - a\right) \equiv Q_{shell}$$

$$\oint \vec{E}\cdot d\vec{a} = 4\pi r^2 E$$, like in the pervious problem

$$4\pi r^2 E = \frac{4\pi k(b-a)}{\varepsilon_0}$$

$$\boxed{\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{4\pi k(b-a)}{r^2}\hat{r} \quad \text{or} \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q_{shell}}{r^2}\hat{r} \quad \text{like a pt. charge!}}$$

(ii) $a < r < b$

$$q_{encl} = \int \rho \, d\tau$$

$$= k \underbrace{\int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi}_{4\pi} \underbrace{\int_{r'=a}^{r} \frac{1}{(r')^2}(r')^2 dr}_{(r-a)}$$

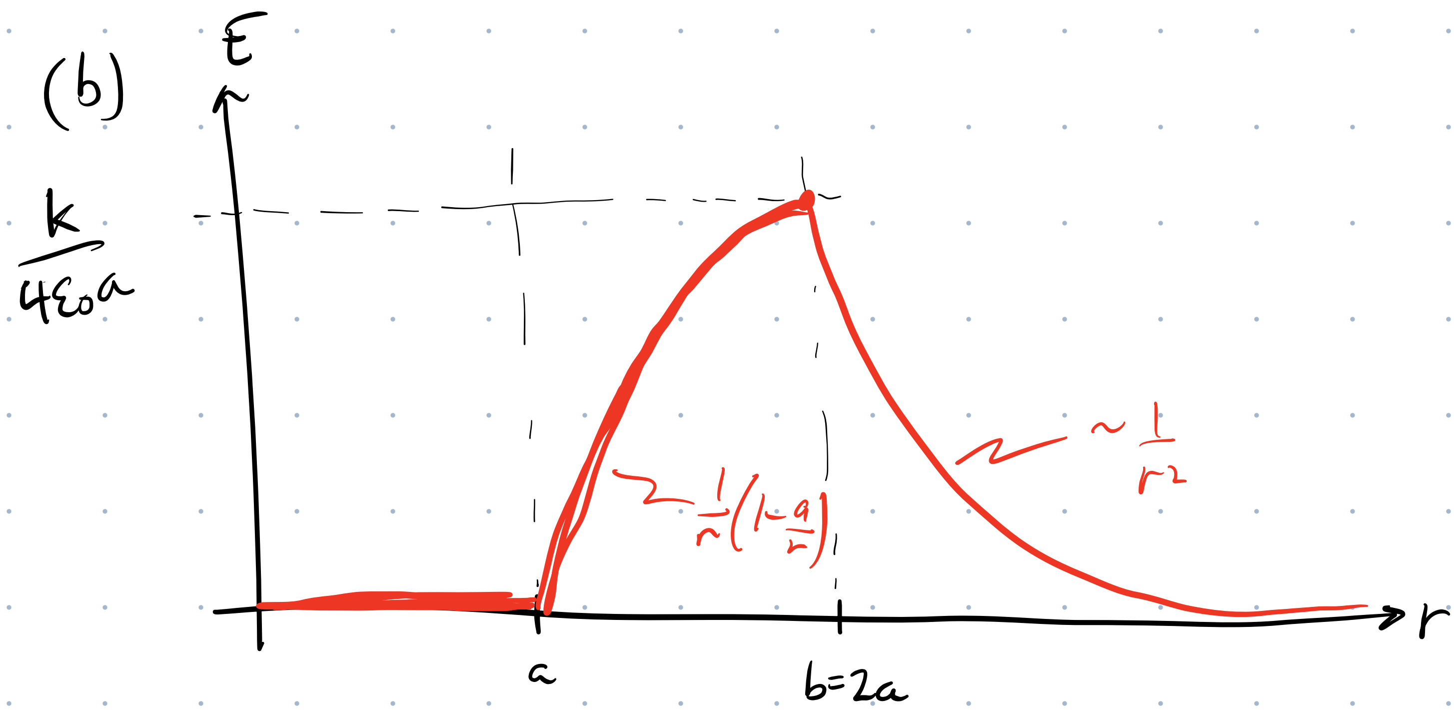
$\therefore q_{encl} = 4\pi k(r-a)$

Again, $\oint \vec{E} \cdot d\vec{a} = 4\pi r^2 E$

$$\therefore 4\pi r^2 E = \frac{4\pi k(r-a)}{\varepsilon_0}$$

$$\boxed{\begin{aligned}\therefore \vec{E} &= \frac{1}{4\pi\varepsilon_0} \frac{4\pi k(r-a)}{r^2}\,\hat{r} \\ &= \frac{k}{\varepsilon_0}\frac{1}{r}\left(1-\frac{a}{r}\right)\hat{r}\end{aligned}}$$

(b)



When $r = a$, $E = 0$

$r = 2a$, $E = \frac{k}{\varepsilon_0}\frac{1}{2a}\left(1 - \frac{1}{2}\right) = \frac{k}{4\varepsilon_0 a}$

or $E = \frac{1}{\cancel{4\pi}\varepsilon_0}\frac{\cancel{4\pi}k(2a-a)}{(2a)^2} = \frac{k\,a}{\varepsilon_0\,4a^2}$

$= \frac{k}{4\varepsilon_0 a}$ ✓

(same as above)

(i) $r < a$ $E = 0$

(ii) $a < r < b$ $E \propto \frac{1}{r}\left(1 - \frac{a}{r}\right)$

(iii) $r > b$ $E \propto \frac{1}{r^2}$